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Lacunary Differentiability of Functions in $\mathbb{R}^n$

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The existence of directional derivatives of functions in the Sobolev spaces $L_k^p(\mathbb{R}^n)$ is studied. The novelty consists in calculating them through lacunary incremental quotients. Under these conditions no restrictions on p are necessary; the condition $p > (n/k)$ can be dropped.

0. INTRODUCTION

Several authors have studied the existence of differentials in the Stolz sense (total differentials) of functions in the Sobolev spaces $L_k^p(\mathbb{R}^n)$. (Here we use the notation of [7].) There are well-known classical results in this direction. Namely: If $f \in L_k^p(\mathbb{R}^n)$ and $p > (n/k)$, $1 \leqslant k \leqslant n$, then f possesses a total differential of order k almost everywhere in $\mathbb{R}^n$. In other words, if $\Delta f(x)$ denotes $f(x+h) - f(x)$ and $\Delta_n^k f(x) = \Delta_h(\Delta_h^{k-1} f(x))$, then

$$\lim_{|h| \to 0} \frac{|\Delta_h^k f(x) - \sum_{|\alpha| = k} h^\alpha D^\alpha f(x)|}{|h|^k} = 0$$

a.e. in $\mathbb{R}^n$.

If $p < n/k$, then one can construct a function in $L_k^p(\mathbb{R}^n)$ that is discontinuous every where. See [2, 5, 7, 9].

The last result was considerably refined in papers [1] and [3].

Paper [1] characterizes the Sobolev–Orlicz classes of functions having the property that all their functions possess a total differential of order 1 almost everywhere in $\mathbb{R}^n$.

Paper [3] extends this to the case $k > 1$. More precisely, we have: Let $\Psi(t)$ be non-negative, continuous, convex and increasing in $t \geqslant 0$. Consider the class Loc $L_k^\Psi(\mathbb{R}^n)$ of functions in $\mathbb{R}^n$ whose distributional derivatives $D^\alpha f$ satisfy

$$\int_{\text{Loc}} \Psi(|D^\alpha f|)\, dx < \infty, \qquad |\alpha| \leqslant k, \tag{0.1}$$

where $1 \leqslant k < n$.

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If

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty, \tag{0.2}$$

then the functions of Loc $L_k^\Psi(\mathbb{R}^n)$ possess a total differential of order k almost every where in $\mathbb{R}^n$.

Conversely, if the integral (0.2) is divergent, then there is a function in Loc $L_k^\Psi(\mathbb{R}^n)$ that is discontinuous everywhere.

An equivalent characterization in terms of Lorentz spaces is discussed briefly in [8]. This characterization is an easy consequence of the relation[1]

$$\operatorname{loc} L_{n/k,1} = \bigcup_{\Psi} \operatorname{loc} L^\Psi(\mathbb{R}^n)$$

where

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty.$$

Here $L_{n/k,1}$ stands for the usual Lorentz space with parameters n/k and 1.

The aim of this paper is to study the existence of directional derivatives of functions in $L_k^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant n/k$, $1 \leqslant k \leqslant n$.

We shall prove the existence a.e. of directional derivatives when the incremental quotient is evaluated through lacunary increments. One of the typical results is that if $f \in L_1^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant n$, and β is a fixed vector in the unit sphere of $\mathbb{R}^n$, then the limit

$$\lim_{k\to\infty} 2^k\{f(x+2^{-k}\beta)-f(x)\} \tag{0.3}$$

exists a.e. in $\mathbb{R}^n$. Here k runs through the natural numbers.

This result is surprising considering the fact that f can be chosen to be discontinuous everywhere.

When calculating partial or directional derivatives through lacunary incremental quotients, the restriction on p ($p > n/k$) becomes unnecessary.

All the usual properties valid for functions in $C^k(\mathbb{R}^n)$ are recovered if one thinks in a.e. terms.

We shall discuss the main results in the next section.

[1] See Appendix.

1. Statement and Proof of the Main Results

We define $\Delta_{\beta,l} f(x) = \Delta^{(l)}_{\beta,l} f(x) = f(x + 2^{-l}\beta) - f(x)$, where β is a fixed vector in the unit sphere of $\mathbb{R}^n$ and l runs through the natural numbers. Likewise, we define

$$\Delta^{(m)}_{\beta,l} f(x) = \Delta_{\beta,l}(\Delta^{(m-1)}_{\beta,l} f(x)). \tag{1.1}$$

We introduce the maximal dyadic derivative

$$f^{*(m)}_{\beta}(x) = \sup_{l} 2^{lm} \left|\Delta^{(m)}_{\beta,l} f(x)\right|. \tag{1.2}$$

The dyadic directional derivative of order m of f at x in the direction of β is

$$\lim_{l \to \infty} 2^{lm} \Delta^{(m)}_{\beta,l} f(x) = D^m_{\beta} f(x). \tag{1.3}$$

THEOREM A. *Let $f \in L^p_k(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$. Then*:

(i) $\|f^{*(k)}_{\beta}\|_p \leqslant C_p \|f\|_{p,k}$, $p > 1$.

(ii) meas $\{x : f^{*(k)}_{\beta}(x) > \lambda\} < C_1/\lambda \, \|f\|_{1,k}$ $\forall \lambda > 0$.

(iii) *If* $1 \leqslant p \leqslant \infty$, *then* $D^k_{\beta} f$ *exists a.e.*

Here $\|f\|_{p,k} = \sum_{0 \leqslant |\alpha| \leqslant k} \|D^{\alpha} f\|_p$, where $D^{\alpha} f$ denotes, as usual, a distributional derivative.

Proof. Following standard arguments it is enough to prove ((i) and (ii). Without loss of generality we may assume that $f \in C^{\infty}_0(\mathbb{R}^n)$ and consider the Taylor expansion of f about the point z:

$$\begin{aligned} f(s) = &\sum_{0 \leqslant |\alpha| \leqslant k-1} \frac{D^{\alpha} f(z)}{\alpha!} (s - z)^{\alpha} \\ &+ k \sum_{|\alpha| = k} (s - z)^{\alpha} \int_0^1 t^{k-1} D^{\alpha} f(s + t(z - s))\, dt. \end{aligned} \tag{1.4}$$

Consider now $\Delta^{(k)}_{\beta,l} f(x)$ and notice that

$$\Delta^{(k)}_{\beta,l} \left\{ \sum_{|\alpha| \leqslant k-1} \frac{D^{\alpha} f(z)}{\alpha!} (x - z)^{\alpha} \right\} = 0. \tag{1.5}$$

Set $x_0 = x$, $x_j = x + j 2^{-l}\beta$, $1 \leqslant j \leqslant k$. Then:

$$|\Delta^{(k)}_{\beta,l} f(x)| \leqslant C \sum_{j=0}^{k} \sum_{|\alpha| = k} \int_0^{|z - x_j|} \rho^{k-1} \left| D^{\alpha} f\left(x_j + \rho \frac{z - x_j}{|z - x_j|}\right) d\rho \right|. \tag{1.6}$$

Integrating with respect to z over $|z-x|<3n2^{-l}$, we obtain

$$|\Delta_{\beta,l}^{(k)} f(x)| \leqslant C2^{nl} \sum_{k=0}^{k} \sum_{|\alpha|=k} \int_{|x-z|<3n2^{-l}} |x-z|^{-n+k} |D^\alpha f(x_j-z)|\, dz. \qquad (1.7)$$

The above constant C depends on the dimension only. Let $K(x)$ be $|x|^{k-n}$ if $|x| \leqslant 6n$ and zero otherwise. Let us introduce the auxiliary kernels $K_j(x)=K(x-j\beta)$, $j=0, 1, 2,..., k$.

From the very definition we see that the kernels $K_j(x)$ are monotone functions of the distance from x to the fixed points $j\beta$, respectively.

Also the kernels $K_j(x)$ belong to $L \log^+ L$ over their support.

An application of Lemmas 1.3 and 1.4 in [4] completes the proof on account of the estimate

$$|2^{kl}\Delta_{\beta,l}^{(k)} f(x)| \leqslant C \sum_{|\alpha|=k} \sum_{j=0} \int 2^{nl} K_j(2^l(x-y)) |D^\alpha f(y)|\, dy. \qquad (1.8)$$

2. Related Results

The existence of mixed derivatives is an easy consequence of Theorem A provided the derivatives are calculated through lacunary increments. Namely:

$$D^2_{\beta_1\beta_2} = D^1_{\beta_2}(D^1_{\beta_1} f).$$

In the case of Bessel Potential Spaces $\mathscr{L}^p_\alpha(\mathbb{R}^n)$ (using the notation of [7] we define

$$f_\beta^{*(\alpha)}(x) = \sup_l 2^{\alpha l} |\Delta_{\beta,l}^{(k)} f(x)|,$$

$$f^{**(\alpha)}(x) = \sup_\beta \sup_l 2^{\alpha l} |\Delta_{\beta,l}^{(k)} f(x)|, \qquad (2.1)$$

where k is the least integer $\geqslant \alpha$; $1 \leqslant \alpha < n$. We have in this case the inequality

$$\|f_\beta^{*(\alpha)}\|_p < C_p \|f\|_{p,\alpha}, \qquad 1<p<\infty. \qquad (2.2)$$

If $n-1<\alpha<n$, we have

$$\|f^{**(\alpha)}\|_p < C_p \|f\|_{p,\alpha}, \qquad 1<p<\infty. \qquad (2.3)$$

$\|f\|_{p,\alpha}$ denotes the Bessel Potential norm introduced in [7].

The proofs of (2.2) and (2.3) are an easy consequence of the representation of Bessel Potentials, using minor modifications of Lemmas 1.3 and 1.4 of [4].

APPENDIX

Proof of

$$\operatorname{loc} L_{n/k,1} = \bigcup_{\Psi} \operatorname{loc} L^{\Psi}(\mathbb{R}^n), \tag{A1}$$

where

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{k/(n-k)} dt < \infty.$$

An equivalent result is

THEOREM. *Let $g(r)$ be positive, decreasing and continuous in* $(0, \infty)$. *If* $0 < \alpha < n$, *then*

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty \tag{A2}$$

if and only if there exists a convex $\Psi(t) \geqslant 0$, $t > 0$, such that

$$\int_0^1 \Psi(g)\, r^{n-1}\, dr < \infty, \tag{A3}$$

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{\alpha/(n-\alpha)} dt < \infty.$$

Proof. Let $g(r)$ be continuous for $r > 0$, decreasing and such that

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty. \tag{A4}$$

Let $\theta(s)$ be a positive convex function such that

$$\theta(r^{\alpha-n}) \sim g(r)\, r^{\alpha-n} \qquad \text{for} \quad r \to 0^+ \tag{A5}$$

(namely, $g(r)\, r^{\alpha-n}/\theta(r^{\alpha-n})$ and its reciprocal are bounded as $r \to 0^+$).
Clearly we have

$$\int_0^1 \theta(r^{\alpha-n})\, r^{n-1}\, dr < \infty. \tag{A6}$$

If Ψ is the conjugate of θ in the Orlicz sense, then

$$\int_1^\infty \left[\frac{t}{\Psi(t)}\right]^{\alpha/(n-\alpha)} dt < \infty \tag{A7}$$

(for details see [3, Lemma e, p. 288]).

We have constructed θ to satisfy

$$\theta(r^{\alpha-n})/r^{\alpha-n} \sim g(r), \qquad r \to 0^+; \tag{A8}$$

but $\theta(s)\, s^{-1} \sim \theta'(s)$, which gives

$$\Psi'(g(r)) \sim r^{\alpha-n}, \qquad r \to 0^+ \tag{A9}$$

(because θ' and Ψ' are inverses of each other). On the one hand we have

$$\Psi'(g(r))\, g(r) \sim r^{\alpha-n} g(r), \qquad r \to 0^+, \tag{A10}$$

while on the other hand

$$\Psi'(s)\, s \sim \Psi(s), \qquad s \to \infty. \tag{A11}$$

Thus

$$\int_0^1 g(r)\, r^{\alpha-n} r^{n-1}\, dr < \infty \tag{A12}$$

implies

$$\int_0^1 \Psi(g)\, r^{n-1}\, dr < \infty.$$

This proves one direction. The other follows from Lemma d in [3, p. 289].

For further details the reader is advised to consult [3].

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